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The formula for the ionization cross-section in the interaction of heavy electrons, obtained earlier by Breit and Tamm (Physical Review, 64, 1943) and by Gurnea (ibid. 69, 1945), is correct in themselves, however, noted a lack of strict in their methods. Although the formulas obtained by them give correct values for the effect, it would be interesting to give a very simple and direct derivation of this formula.

Breit's calculations are based on the following scheme. The field of the nucleus is replaced by the potential

$$\begin{aligned} \psi &= -Ze^2/r \quad (r > r_0) \\ \psi &= -Ze^2/r_0 \quad (r < r_0) \end{aligned} \quad (1)$$

that is, within the nucleus the potential is considered constant.

This corresponds to the introduction of the concept of effective radius - namely, the radius of a sphere with a surface charge, the external field of which coincides with the field of the nucleus.

The levels of an electron in such a field are found by the method of the theory of perturbations. The excitation will be

$$u = -Ze^2(1/r_0 - 1/r) \quad (2)$$

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At $r = 0$, the excitation becomes infinite, and we can, generally speaking, no longer use the ordinary theory of perturbation. In particular, we cannot consider the variation in the energy to be proportional to

$$\int u^2 \psi^2 \cdot dV \quad (2')$$

This is not once clear if we note that V differs from zero only for $r < r_0$, and just in this region does ψ differ strongly from the nonperturbed function.

This difficulty can be overcome if we employ the method of perturbation of the boundary conditions, which method is similar to the method proposed by Froehlich for Schrodinger's equation.

Let us consider the problem outside the nucleus. In this region, obviously, the field in the nonperturbed problem coincides with the real field. The only difference is that at $r = r_0$ the wave function must coincide with the function corresponding to a rectangular hole, while in the real problem the wave function remains a Coulomb one all the way to $r = 0$. Therefore one can consider that in such an "external" problem only the boundary conditions are perturbed at $r = r_0$.

We shall show that the variation in energy can be expressed through the values of the perturbed and nonperturbed wave functions on the surface of the sphere $r = r_0$.

Dirac's equations for the wave functions $\phi_1^0 = \psi_1^0$ and $\phi_2^0 = \psi_2^0$ assume the forms

$$\begin{aligned} \frac{d\phi_1^0}{dy} + \frac{\phi_1^0}{y} &= \frac{1}{2a} \left[1 - \frac{E-V}{\mu} \right] \phi_2^0 \\ \frac{d\phi_2^0}{dy} - \frac{\phi_2^0}{y} &= \frac{1}{2a} \left[1 + \frac{E-V}{\mu} \right] \phi_1^0 \end{aligned} \quad (3)$$

(Note: We employ here the designations of Breit and Rosenthal, mentioned above).

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The corresponding equations for the perturbed functions ϕ_1 and ϕ_2 will be

$$\begin{aligned} \frac{d\phi_1}{dy} + \frac{\phi_1}{y} &= -\frac{1}{2a} \left[1 - \frac{E+V}{\mu} \right] \phi_2 - \frac{Z}{2\mu a} \phi_2 \\ \frac{d\phi_2}{dy} - \frac{\phi_2}{y} &= \frac{1}{2a} \left[1 + \frac{E+V}{\mu} \right] \phi_1 + \frac{Z}{2\mu a} \phi_1 \end{aligned} \quad (1)$$

where μ is the electron's mass; $y = (2\mu e^2/\hbar^2)r = 2ar/a_H$; $a_H = \hbar^2/\mu e^2$; $a = Ze^2/\hbar c$; and e (ion z) is the variation in energy.

Multiplying the first equations (1) and (2) respectively by $\phi_2^0 - \phi_1^0$, $-\phi_2^0$, ϕ_1^0 and adding, we easily obtain

$$\frac{d}{dy} (\phi_2 \phi_1^0 - \phi_1^0 \phi_2) + \frac{Z}{2\mu a} (\phi_1^0 \phi_2 + \phi_2^0 \phi_1) = 0 \quad (2)$$

or converting to the variable r we get

$$\frac{d}{dr} (\phi_2 \phi_1^0 - \phi_1^0 \phi_2) + \frac{Z}{a_H} \frac{1}{r} (\phi_1^0 \phi_2 + \phi_2^0 \phi_1) = 0 \quad (3)$$

Considering that the perturbed wave function at infinity coincides with the unperturbed one (which one can always be obtained by normalization), we can, by integrating (3) from r_0 to infinity ∞ , replace, in the second term, ϕ_1 and ϕ_2 by ϕ_1^0 and ϕ_2^0 and replace the lower limit during integration of this term by zero. By virtue of normalization, to have

$$4\pi \int_0^\infty (\phi_1^0 \phi_1^0 + \phi_2^0 \phi_2^0) dr = 1 \quad (4)$$

and from (6) we obtain

$$\varepsilon = 4\pi a_H \mu c^2 [\phi_2^0(r_0) \phi_1^0(r_0) - \phi_2^0(r_0) \phi_1(r_0)] \quad (5)$$

The values of the functions at $r = r_0$ are computed in the work of Brit and Rosenthal:

$$\left. \begin{aligned} \phi_1^0/a &= C_0 J_{2p}(2y^{\frac{1}{2}}) \\ \phi_2^0 &= -C_0(1+p) J_{2p}(2y^{\frac{1}{2}}) \end{aligned} \right\} \quad (6)$$

$$\left. \begin{aligned} \phi_1/a &= C\phi_1^0/C_0 a + C_- J_{2p}(2y^{\frac{1}{2}}) \\ \phi_2 &= C\phi_2^0/C_0 + C_- J_{2p}(2y^{\frac{1}{2}}) \cdot (1-p) \end{aligned} \right\} \quad (7)$$

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where $p^2 = 1 - (2e^2/\hbar c)^2$; J_{2p} is a Bessel function; and constants C and C_0 are determined by the conditions of continuity at $r = r_0$. C_0 is found from formula (7):

$$C_0 = \psi(0) a_H / 2Z, \quad (11)$$

where $\psi(0)$ is the value of the nonrelativistic wave function at zero.

Substituting (5) and (10) into (6) and converting the Bessel function into a series, we obtain after limiting ourselves to the first term of the expansion of the full order $2p$ function

$$\varepsilon = 8\pi Z e^2 \psi_0^2 \frac{y_0^{2p}}{[I(2p+1)]^2} \cdot \frac{\zeta(r_0)(1+p)+1}{(-1)\zeta(r_0)+1}, \quad (12)$$

where ζ is the difference of the terms of the hypergeometric series, we have

$$\delta\varepsilon = 8\pi Z e^2 C_0^2 \psi_0^2 \frac{y_0^{2p}}{[I(2p+1)]^2} \cdot \frac{\zeta(r_0)(1+p)+1}{\zeta(r_0)(-1)+1} \cdot \frac{\delta\zeta}{\zeta}, \quad (13)$$

where

$$y_0 = \frac{2Z}{a_H} r_0; \quad \zeta(r) = \left(\frac{V}{V(r) - 2mc^2} \right)^{1/2} \frac{J_{1/2}(kr)}{J_{1/2}(kr_0)}, \quad (14)$$

$$V = \frac{1}{\hbar c} \sqrt{V - 2mc^2}^{1/2}.$$

It can be assumed that

$$\zeta(r) \approx -\frac{1}{\alpha} \frac{J_{1/2}(kr)}{J_{1/2}(kr_0)} \approx -\frac{1}{3}. \quad (15)$$

Finally we obtain (in units of cm^{-1}):

$$\delta\varepsilon = \frac{8\pi R a_H^3}{Z^2} \psi_0^2 \frac{p^2}{[I(2p+1)]^2} \cdot \frac{2-p}{2+p} y_0^{2p} \frac{\delta r_0}{r_0}, \quad (16)$$

where R is the Rydberg constant $3.058 \cdot 10^{-6}/\text{cm}$.

Parah's formula assumes the form:

$$\delta\varepsilon_R = \frac{4\pi R a_H^3}{Z} \psi_0^2 \frac{1}{[I(2p+1)]^2} \frac{1+p}{1+2p} y_0^{2p} \frac{\delta r_0}{r_0} \quad (17)$$

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Then the corrected expression differs from the old one by the multiple

$$\xi = \frac{\delta \xi}{\delta \xi_0} = \frac{2\rho^2(1-\rho)(1+2\rho)}{(1+\rho)(2+\rho)}$$

As $\rho \rightarrow 1$, we get $\xi \rightarrow 1$; that is, for that value both formulas coincide. For $\rho = 0.8$, we have $\xi = 0.8$.

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